TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 348, Number 6, June 1996

TENSOR PRODUCTS OVER ABELIAN W^*−ALGEBRAS

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Abstract. Tensor products of C^*−algebras over an abelian W^*−algebra Z are studied. The minimal C^*−norm on $A \odot_Z B$ is shown to be just the quotient of the minimal C^*−norm on $A \odot B$ if A or B is exact.

1. Introduction

The author's initial motivation for studying tensor products of C^*−algebras as modules over central abelian subalgebras was the fact that the Haagerup tensor product of modules (see [1], [6] and [17]) and some tensor products of von Neumann algebras (see [23], [24] and [28]) have nice properties. Later (five months after the first version of this paper was submitted) we were informed[1] that C^*−tensor products over abelian C^*−algebras have been studied recently also by Blanchard ([3], [4]) and Giordano and Mingo [10] (in the present revised version some of the overlapping results are omitted). Moreover such tensor products are closely related to the recent work of Kirchberg and Wassermann [15] and occurred previously also in [14], [8] and [9].

By a *C^*−algebra over an abelian unital C^*−algebra C* we mean a C^*−algebra A together with a unital $*$−homomorphism φ from C into the center of the multiplier algebra $M(A)$ of A. Usually we shall write ca instead of $\varphi(c)a$ for $c \in C$ and $a \in A$. If there is a faithful representation π of $M(A)$ on a Hilbert space such that $\pi\varphi(C)$ is closed in the weak operator topology, then A is called a *proper C^*−algebra over C*. In this case the unique normal extension of $\pi\varphi$ to the universal enveloping W^*−algebra Z of C maps Z onto $\pi\varphi(C)$, hence we can regard A as a C^*−algebra over Z instead of C.

Given C^*−algebras A and B over C, the algebraic tensor product $A \odot_C B$ is by definition the quotient $(A \odot B)/N$, where $A \odot B$ denotes the algebraic tensor product of A and B viewed as complex algebras and N is the linear span of all elements of the form $ac \otimes b - a \otimes cb$ ($a \in A$, $b \in B$, $c \in C$). Since by assumption C commutes with A and B, the subspace N is a self-adjoint two-sided ideal in $A \odot B$; hence $A \odot_C B$ is an algebra (over C) with an involution. It turns out that most of the basic theory of tensor products of C^*−algebras (see [13], [19], [26], [7]) can be extended to the context of proper C^*−algebras over C. In fact, some basic results can be extended to the setting of general C^*−algebras over C (see [4]), but proper

Received by the editors March 20, 1995.
1991 *Mathematics Subject Classification.* Primary 46L05; Secondary 46L10.
Key words and phrases. Tensor product, C^*−norm, Hilbert module.

[1]The author is grateful to S. Wassermann and D. Somerset for bringing the works of Blanchard, Giordano and Mingo to his attention, and to E. Blanchard, J. Mingo and S. Wassermann for the preprints.

C*−algebras over C can be regarded as C*−algebras over $Z = C''$ and represented faithfully on self-dual Hilbert Z−modules, which can play the role of Hilbert spaces.

In the preliminary Section 2 of this paper we recall some basic theory of Hilbert modules in a concrete form suitable for our application. Then in Section 3 we study algebraic properties of the product $A \odot_Z B$ and introduce the *spatial* norm on $A \odot_Z B$ by faithfully representing A and B on some Hilbert Z−modules E and F and then regarding $A \odot_Z B$ as an algebra of operators on $E \otimes_Z F$. In the case A or B is exact we show that the kernel of the natural map from $A \otimes B$ to $A \otimes_Z B$ (= the completion of $A \odot_Z B$ in the spatial norm) is just the closure of the ideal N defined above. If A and B are not proper, then it can happen that there is no C*−norm on $A \odot_C B$ (see Example 3.9). This is essentially an answer to the question of Elliott [8, p. 48], which has been solved recently and independently also by Blanchard [4]. In Section 4 we derive a formula for the spatial tensor norm which is analogous to the well known classical one (in the case $Z = \mathbb{C}$), but using Z−valued states instead of usual scalar valued states.

Throughout this paper C will denote a commutative unital C*−algebra, Z a commutative von Neumann algebra (usually the universal W*−envelope of C) and A, B will be general C*−algebras over C or Z.

2. Preliminaries concerning Hilbert C*−modules

A (*right*) *Hilbert module over a C*−algebra* A is a right A−module E equipped with an A−valued inner product $\langle\cdot,\cdot\rangle$, which is linear over A in the second factor, such that, with the norm $\|x\| \stackrel{\text{def}}{=} \|\langle x,x\rangle\|^{1/2}$ $(x \in E)$, E is a Banach space (see [16], [20] and [21]). A Hilbert A−module E is *self-dual* if each bounded A−module homomorphism $\rho : E \to A$ is of the form $\rho(x) = \langle x_0, x\rangle$ for some $x_0 \in E$.

Given a Hilbert A−module E, we denote by $L(E)$ the C*−algebra of all operators on E that have an adjoint (each such operator is automatically a bounded homomorphism of the A−module E). The *linking algebra*, $\Lambda(E)$, of E is the C*−subalgebra of $L(E \oplus A)$ consisting of all matrices of the form

$$\begin{bmatrix} b & x \\ y^* & a \end{bmatrix} \quad (a \in A,\ x,y \in E,\ b \in L(E)).$$

The concept of linking algebra was introduced in [5], but with $L(E)$ replaced by the subalgebra of 'compact' operators. We shall regard A, E and $L \stackrel{\text{def}}{=} L(E)$ as subsets of $\Lambda(E)$ in the obvious way:

$$A \cong \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix}, \quad E \cong \begin{bmatrix} 0 & E \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad L \cong \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix}.$$

Then the inner product $\langle x,y\rangle$ in E becomes simply the product x^*y in $\Lambda(E)$ and the module multiplication becomes a part of the internal multiplication in $\Lambda(E)$.

A Hilbert module E over a von Neumann algebra R is *faithful* if the ideal $\langle E,E\rangle$ is weak* dense in R. Equivalently, E is faithful iff $Ea = 0$ implies $a = 0$ $(a \in R)$.

From the results of Paschke [20] and Rieffel [21] one can deduce the following concrete realization of Hilbert modules.

Theorem 2.1. *Let E be a faithful Hilbert module over a von Neumann algebra $R \subseteq B(\mathcal{H})$ and let Z be the center of R. Then there exists a von Neumann algebra $\overline{\Lambda}$ and two projections $p, q \in \overline{\Lambda}$ such that $R = q\overline{\Lambda}q$, E is contained in $\overline{E} = p\overline{\Lambda}q$ as a weak* dense Hilbert R−submodule and $L(E)$ is weak* dense in $L(\overline{E}) = p\overline{\Lambda}p$,* *where the $R-$valued inner product in $\overline{E}$ is defined by $\langle x, y\rangle = x^*y$. Moreover, each bounded $R-$module homomorphism $\rho : E \to R$ is of the form $\rho(y) = \langle y_0, y\rangle$ for some $y_0 \in \overline{E}$. In particular, E is self-dual iff $\overline{E} = E$, and in this case $L(E) = L(\overline{E})$ is a W^*-algebra and $\Lambda(E) = \overline{\Lambda}$.*

Moreover, if $R = Z$ (that is, if R is abelian) and E is self-dual, then E is (isometrically) isomorphic to a Hilbert module of the form Le, where $L = L(E)$ is realized as a von Neumann algebra on some Hilbert space such that its commutant L' is isomorphic to Z and e is an abelian projection with central carrier $c(e) = 1$ in L. Here the inner product in E is defined by $\langle x, y\rangle = \tau^{-1}(x^*y)$, where τ is the isomorphism from Z onto $Ze = eLe$ given by $\tau(z) = ze$.

The first part of Theorem 2.1 can be proved easily also by extending the identity representation of R to a representation of $\Lambda(E)$ on some Hilbert space $\mathcal{L} \supseteq \mathcal{H}$, and more details of this proof will appear in the expository article [18]. To prove the last part of the theorem, we may then assume that $E = p\Lambda q$, where Λ is a von Neumann algebra and the central carriers of the projections p and q in Λ may be assumed to be 1 (otherwise reduce Λ). Then, since $q\Lambda q = Z$ is abelian, it follows that q is equivalent to a projection $e \le p$ in Λ. This implies that E is isomorphic to $p\Lambda e = Le$, since $L = p\Lambda p$.

From Theorem 2.1 it follows in particular that a self-dual Hilbert module E over a von Neumann algebra is a dual Banach space (this was first proved in [20]). Note that the weak* topology on E can be defined intrinsically (without using any representations): on bounded sets it is given by the family of seminorms $x \mapsto \rho(\langle y, x\rangle)$, where $y \in E$ and ρ is a normal functional on R. From Theorem 2.1 we see also that each Hilbert $R-$module can be embedded into a self-dual Hilbert $R-$module $\overline{E}$ such that E is weak* dense in $\overline{E}$ and each $b \in L(E)$ extends uniquely to an operator in $L(\overline{E})$ (all this was also proved in [20]). $\overline{E}$ will be called *the self-dual hull of E*.

3. The spatial tensor product

If E and F are Hilbert modules over C, then (regarding F as a left $C-$module by $cy \stackrel{\text{def}}{=} yc$ for $y \in F$ and $c \in C$) the algebraic tensor product $E \odot_C F$ can be equipped with the $C-$valued inner product by

$$\langle x \otimes_C y, u \otimes_C v\rangle = \langle x, u\rangle\langle y, v\rangle \qquad (x, u \in E,\ y, v \in F).$$

Remark 3.1. This construction is just a special case of the internal tensor product of Hilbert modules over not necessarily abelian C$^*-$algebras (see Chapter 4 in [16]). In particular, it follows from Proposition 4.5 in [16] that the above inner product is indeed positive definite.

The completion of $E \odot_C F$ in the norm induced from the inner product is a Hilbert module over C and will be denoted by $E \otimes_C F$. If E and F are Hilbert modules over an abelian W$^*-$algebra Z, then the self-dual hull of $E \otimes_Z F$ will be denoted by $E\overline{\otimes}_Z F$.

Using the techniques from [16, Chapter 4] it is easy to see that for each $a \in L(E)$ and $b \in L(F)$ there exists a unique operator $a \otimes_C b \in L(E \otimes_C F)$ such that $(a \otimes_C b)(x \otimes_C y) = ax \otimes_C by$ for all $x \in E$ and $y \in F$, and that $\|a \otimes_C b\| \le \|a\|\|b\|$.

We denote by $\mathrm{M}_n(E)$ the set of all $n \times n$ matrices with entries in a given set E. The following lemma is formally similar to Theorem 5.5.4 in [13] and is equivalent to Theorem 3.1 of [10]. We shall give a direct proof.

Lemma 3.2. *Let E and F be Hilbert modules over an abelian W^*-algebra Z and $a_i \in L(E)$, $b_i \in L(F)$ $(i = 1, \dots, n)$. Then the identity $\sum_{i=1}^n a_i \otimes_Z b_i = 0$ holds in $L(E \otimes_Z F)$ if and only if there exists a projection $p = [p_{ij}] \in \mathrm{M}_n(Z)$ such that*

$$\sum_{j=1}^n a_j p_{ji} = 0 \quad \textit{and} \quad \sum_{j=1}^n p_{ij} b_j = b_i$$

for all $i = 1, \dots, n$.

Proof. For arbitrary $x \in E$, $y \in F$ put

$$\mathbf{x} = \begin{bmatrix} a_1x & \dots & a_nx \\ 0 & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} b_1y & \dots & b_ny \\ 0 & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & 0 \end{bmatrix}$$

and regard $\mathrm{M}_n(E)$ and $\mathrm{M}_n(F)$ as subspaces in $\mathrm{M}_n(\Lambda(E))$ and $\mathrm{M}_n(\Lambda(F))$ (respectively). Let $a_x = [\langle a_i x, a_j x\rangle] = \mathbf{x}^*\mathbf{x}$ and let $b_y = [\langle b_j y, b_i y\rangle]$ be the transpose of the matrix $\mathbf{y}^*\mathbf{y}$ in $\mathrm{M}_n(Z)$. Suppose that $w \stackrel{\text{def}}{=} \sum_{i=1}^n a_i \otimes_Z b_i = 0$. Then we have

$$\sum_{i,j=1}^n \langle a_i x, a_j x\rangle \langle b_i y, b_j y\rangle = \langle w(x \otimes_Z y), w(x \otimes_Z y)\rangle = 0. \tag{3.1}$$

Since a_x is positive, there exists a positive $g = [g_{ij}] \in \mathrm{M}_n(Z)$ such that $a_x = g^2$, and (3.1) can be rewritten as

$$0 = \sum_{i,j=1}^n \sum_{k=1}^n g_{ik} g_{kj} (b_y)_{ji} = \sum_{k=1}^n (g b_y g)_{kk}.$$

Thus all the diagonal entries of the positive operator matrix gb_yg are 0 (note that b_y is positive as the transpose of the positive matrix $\mathbf{y}^*\mathbf{y}$ over the abelian C^*-algebra Z); hence $gb_yg = 0$, which implies that

$$a_x b_y = 0. \tag{3.2}$$

Let $p_y \in \mathrm{M}_n(Z)$ be the range projection of b_y. Then from (3.2) we have $a_x p_y = 0$ and $p_y^\perp b_y = 0$ (where $p_y^\perp = 1 - p_y$), which implies (by a simple computation, writing p_y as a matrix and using the definiteness of the inner product) that

$$[a_1x, \dots, a_nx] p_y = 0 \quad \text{and} \quad p_y^\perp \begin{bmatrix} b_1y \\ \vdots \\ b_ny \end{bmatrix} = 0.$$

Hence, with $p = \bigvee_{y \in F} p_y$, we have

$$[a_1, \dots, a_n] p = 0 \quad \text{and} \quad p \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}.$$

With $[p_{ij}] = p$ $(p_{ij} \in Z)$, we now see that $\sum_{i=1}^n a_i \otimes_Z b_i = \sum_{i=1}^n a_i \otimes_Z \sum_{j=1}^n p_{ij} b_j = \sum_{j=1}^n \sum_{i=1}^n a_i p_{ij} \otimes_Z b_j = 0$. This proves the lemma in one direction. The proof of the converse is trivial. □

The following result is an immediate consequence of Lemma 3.2.

Corollary 3.3. *If $A \subseteq L(E)$ and $B \subseteq L(F)$ are C*−algebras over Z, then the natural map $\Theta : A \odot_Z B \to L(E \otimes_Z F)$ is one to one. Thus, if $A_0 \subseteq A$ and $B_0 \subseteq B$ are C*−algebras over Z, then the natural map $A_0 \odot_Z B_0 \to A \odot_Z B$ is also one to one.*

To see that Corollary 3.3 cannot be extended to general abelian C*−algebras in place of Z, let $C \subseteq l^\infty$ be the C*−algebra of all convergent sequences and let E consists of all sequences (c_i) with the entries in C such that $\sum_i c_i^* c_i$ converges in norm. Then E is a Hilbert C module (the direct sum of countably many copies of C) and l^∞ can be regarded as a C*−subalgebra of $L(E)$ by the diagonal action $(\alpha_i)(c_i) = (\alpha_i c_i)((\alpha_i) \in l^\infty, (c_i) \in E)$. Let $a = (1, 0, 1, 0, \dots)$ and $b = (0, 1, 0, 1, \dots)$ and let $e_1, e_2, \dots$ be the standard basic elements of E. Note that $e_i \otimes_C e_j = 0$ if $i \neq j$ and $(a \otimes_C b)(e_i \otimes_C e_j) = 0$ for all i, j. Thus $a \otimes_C b = 0$ in $L(E \otimes_C E)$. On the other hand, $a \otimes_C b \neq 0$ in $l^\infty \odot_C l^\infty$. Indeed, if ϕ and ψ are characters on l^∞ annihilating the sequences converging to 0 (hence $\phi|C = \psi|C$) such that $\phi(a) = 1 = \psi(b)$, then the correspondence $x \otimes_C y \mapsto \phi(x)\psi(y)$ can be extended to a linear map $\rho : l^\infty \odot_C l^\infty \to C$ such that $\rho(a \otimes_C b) = 1$.

Remark 3.4. It follows from [14, 1.5] that each C*−algebra A over a C*−algebra C can be embedded into $L(E)$ for some self-dual Hilbert Z−module E, where Z is the universal von Neumann envelope of C. To see this more directly, we may assume that $A \subseteq B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ such that $[A\mathcal{H}] = \mathcal{H}$. Let φ be the $*$−homomorphism from C into the center of $M(A) \subseteq B(\mathcal{H})$, which determines the C−module structure on A, let $\overline{\varphi}$ be the normal extension of φ to Z and let $p \in Z$ be the projection such that $\overline{\varphi}|Zp$ is an isomorphism onto $\overline{\varphi}(Z)$. Put $Z_0 = \overline{\varphi}(Z)$ and $L = Z_0'$. Let e be any abelian projection in L with central carrier 1 and let $E \stackrel{\text{def}}{=} Le$ have the Z_0−valued inner product defined as in the last sentence of Theorem 2.1. Then $L(E) = L$; hence $A \subseteq L(E)$. Since $Z_0 \cong Zp$, E can be regarded as a Hilbert Z−module, and by adding the direct summand $Zp^\perp$ to E we obtain a faithful self-dual Hilbert Z−module.

If A and B are proper C*−algebras over C and Z is the universal W*−envelope of C, then by Remark 3.4 we may assume that $A \subseteq L(E)$ and $B \subseteq L(F)$ for some (faithful self-dual) Hilbert Z−modules E and F, and $A \odot_Z B$ can be regarded as a subalgebra of $L(E \otimes_Z F)$ by Corollary 3.3 and therefore inherits a C*−norm, which will be called *the spatial norm* on $A \odot_Z B$ $(= A \odot_C B)$. We shall obtain a formula for this norm which will be independent of the choice of embedding of A and B into operators on Hilbert Z−modules. The completion of $A \odot_Z B$ in the spatial norm will be denoted by $A \otimes_Z B$ (or $A \otimes_C B$) and called the *spatial tensor product of A and B*. More generally, we denote by $A \otimes_Z^\alpha B$ the completion of $A \odot_Z B$ in a given C*−norm α.

Remark 3.5. The theory of such tensor products can be reduced to the case where Z is contained in the centers of A and B. To see this, we may assume that $A \subseteq B(\mathcal{H})$ and $B \subseteq B(\mathcal{K})$ and let φ, ψ be the normal $*$−homomorphisms from Z into the centers of the multiplier algebras $M(A)$ and $M(B)$ (resp.) that determine the Z−module structure on A and B. As in the classical case (see [27, T.6.1]) there is a natural $*$−homomorphism from $M(A) \odot_Z M(B)$ into $M(A \otimes_Z^\alpha B)$. This homomorphism is injective, since a simple argument (somewhat similar to the one in the proof of Lemma 3.2) shows that an element w of $M(A) \odot_Z M(B)$ annihilates all $x \otimes_Z y \in A \odot_Z B$ only if $w = 0$. Consequently, we may regard $M(A) \odot_Z M(B)$

as a subalgebra of $M(A \otimes_Z^\alpha B)$, from which $M(A) \odot_Z M(B)$ inherits a norm which extends the norm α on $A \odot_Z B$. This shows that in studying C$^*-$norms there is no essential loss of generality in restriction to unital algebras. Then φ and ψ are normal $*-$homomorphisms from Z into the centers of A and B (resp.), hence $\ker\varphi = e^\perp Z$ and $\ker\psi = f^\perp Z$ for some projections $e, f \in Z$. Put $p = ef$. Then $A \odot_Z B = Ap \odot_{Zp} pB$; hence, replacing A, B and Z with Ap, Bp and Zp, we may assume that φ and ψ are $*-$isomorphisms. Moreover, the decomposition of $\varphi(Z)'$ and $\psi(Z)'$ into homogeneous parts (see [13] or [26]) induces the decomposition of $A \odot_Z B$, since $\varphi(Z)$ and $\psi(Z)$ are contained in centers of A and B. It suffices to study each summand separately. Finally, replacing A, B and Z by suitable isomorphic algebras (of the form $A \otimes 1, \dots$), we may assume that Z is contained in the centers of A and B and that Z' is homogeneous.

For a unital abelian C$^*-$algebra C we shall always denote by Δ the spectrum of C (= the space of pure states) and identify C with the C$^*-$algebra of all continuous functions on Δ. For each $s \in \Delta$ we denote by C_s the kernel of s. If C is contained in the center of a C$^*-$algebra A, then the closed ideal of A generated by C_s will be denoted by A_s and called the *Glimm ideal at s*. It is well known that A_s is the intersection of left kernels of all pure states ρ on A satisfying $\rho(C_s) = 0$ and that

$$\|a\| = \sup_{s\in\Delta} \|a(s)\| \tag{3.3}$$

for each $a \in A$, where $a(s)$ denotes the coset of a in $A(s) \overset{\text{def}}{=} A/A_s$. This implies that the natural $*-$homomorphism

$$\Psi_A : A \to \bigoplus_{s\in\Delta} A(s), \quad \Psi_A(a)(s) = a(s)$$

is one to one. For each $a \in A$ the function $s \mapsto \|a(s)\|$ is upper semi-continuous and, if C is a W$^*-$algebra, then this function is continuous. All this was proved by Glimm in [11, Section 4] for the case C is the center of A, but the same arguments apply if C is a C$^*-$subalgebra of the center. (To prove the continuity of the function $s \mapsto \|a(s)\|$ in the case C is a von Neumann algebra, regard A as a subalgebra of C' and use [11, Lemma 10] and Remark 3.6 below.)

Remark 3.6. If B is a C$^*-$subalgebra of A containing C, then $B_s = A_s \cap B$ for each $s \in \Delta$, since each pure state on B extends to a pure state on A. It follows that the natural $*-$homomorphism $B(s) \to A(s)$ is injective; hence $B(s)$ will be regarded as a C$^*-$subalgebra of $A(s)$.

If C is contained in the centers of two C$^*-$algebras $A, B \subseteq B(\mathcal{H})$, then the two quotient mappings $\alpha_s : A \to A(s)$ and $\beta_s : B \to B(s)$ induce a map $\iota_s : A \odot_C B \to A(s) \odot B(s)$ (where $\odot$ means $\odot_{\mathbb{C}}$), and the collection of all maps ι_s $(s \in \Delta)$ defines a $*-$homomorphism

$$\iota : A \odot_C B \to \bigoplus_{s\in\Delta} A(s) \otimes B(s), \quad \iota(x)(s) = \iota_s(x). \tag{3.4}$$

Given a Hilbert $C-$module E and $s \in \Delta$, let

$$E_s = \{x \in E : \ \langle x, x\rangle(s) = 0\}.$$

Then E_s is a Hilbert submodule of E and the quotient $E(s) = E/E_s$ is a Hilbert space with the inner product

$$\langle x(s), y(s)\rangle = \langle x, y\rangle(s),$$

where $x(s)$ and $y(s)$ denote the cosets of x and y in $E(s)$. (The completeness of $E(s)$ follows since the norm $\|x(s)\|$ in $E(s)$ induced from the inner product coincides with the quotient norm as can be seen by a standard application of an approximate unit of C_s, but this will not be important here.) There is a representation π_s of $L(E)$ on $E(s)$ defined by $\pi_s(a)x(s) = (ax)(s)$.

Proposition 3.7. *The $*-$homomorphism*

$$\pi : L(E) \to \bigoplus_{s\in\Delta} B(E(s)), \quad \pi(a)(s) = \pi_s(a)$$

is one to one. If $C = Z$ is an abelian W^*-algebra and E is faithful and self-dual, then the kernel of π_s is the Glimm ideal $L(E)_s$ for each $s \in \Delta$.

Proof. If $a \in \ker\pi$, then $\langle ax, ax\rangle(s) = 0$ for all $x \in E$ and $s \in \Delta$, hence $ax = 0$ and $a = 0$.

Assume now that $C = Z$ is a W^*-algebra and E is self-dual and faithful. Put $L = L(E)$. Then by Theorem 2.1 we may assume that $E = Le$ for some abelian projection $e \in L$ with the central carrier $c(e) = 1$, and we may identify L' with Z (in some representation of L on a Hilbert space). Since the function $t \mapsto \|e(t)\|$ (where $e(t)$ denotes the coset of e in L/L_t, with L_t the Glimm ideal of L at t) is continuous on Δ and can take the values 1 or 0 only, while $c(e) = 1$, it follows that $\|e(t)\| = 1$ for all $t \in \Delta$. Given $x \in L$ and $s \in \Delta$, we then have by the definition of the inner product in $E = Le$ that $\langle xe, xe\rangle(s) = \tau^{-1}(ex^*xe)(s)$, where $\tau : Z \to Ze$ is the $*-$isomorphism $z \mapsto ze$. Since $e(s) \neq 0$ and $z(s) \in \mathbb{C}$ for each $z \in Z$, it follows easily that $\langle xe, xe\rangle(s) = 0$ if and only if $xe \in L_s$; thus $E_s = Le \cap L_s = L_s e$. Given $a \in L$, it follows that $a \in \ker\pi_s$ if and only if $aLe \subseteq L_s$. Since L_s is a prime ideal by [11, Lemma 11] and $e \notin L_s$ (for $\|e(s)\| = 1$), we see that $aLe \subseteq L_s$ if and only if $a \in L_s$. Thus, $\ker\pi_s = L_s$. ◻

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Corollary 3.8. *If Z is contained in the centers of C^*-algebras $A, B \subseteq B(\mathcal{H})$, then the $*-$homomorphism $\iota : A \odot_Z B \to \bigoplus_{s\in\Delta} A(s) \otimes B(s)$ (defined by (3.4) is isometric if $A \odot_Z B$ is equipped with the spatial C^*-norm.*

Proof. We may assume that $A \subseteq L(E)$ and $B \subseteq L(F)$ for some faithful self-dual Hilbert $Z-$modules E and F (see Remark 3.4). It is easy to see that for each $s \in \Delta$ the natural map $E \odot_Z F \to E(s) \otimes F(s)$ induces an (isometric) isomorphism of Hilbert spaces $(E \otimes_Z F)(s) \to E(s) \otimes F(s)$, hence we shall identify these two Hilbert spaces. The representation π_s of $L(E)$ on $E(s)$ induces a $*-$homomorphism $\alpha(s) : A(s) \to B(E(s))$ (which is in fact injective by Proposition 3.7 and Remark 3.6) and in the same way we have a $*-$homomorphism $\beta(s) : B(s) \to B(F(s))$. Let $\psi(s) = \alpha(s) \otimes \beta(s)$ and consider now the commutative diagram

$$\begin{tikzcd}
A \odot_Z B \arrow[r, "\Theta"] \arrow[drr, "\iota"] & L(E \otimes_Z F) \arrow[r, "\pi"] & \bigoplus_{s\in\Delta} B(E(s) \otimes F(s)) \\
& & \bigoplus_{s\in\Delta} A(s) \otimes B(s) \arrow[u, "\psi"]
\end{tikzcd}$$

where Θ is the injection from Corollary 3.3 (and isometric by the definition of the spatial norm), π is the $*-$monomorphism from Proposition 3.7 (with $E \otimes_Z F$

instead of E) and ψ is the direct sum of the $*-$homomorphisms $\psi(s)$. Since ψ, π and Θ are isometric, it follows that ι is isometric. $\square$

For a result related to Corollary 3.8 in a more general context, see [3, 3.21] and [4, 3.1].

Example 3.9. For a general abelian (unital) C^*-algebra C the above $*-$homomorphism ι is not always injective. (This answers a question of Elliott [8, p. 48].) Namely, let A be the C^*-algebra of all bounded functions on the circle $\Delta = [0,1]/\{0,1\}$ and let C be the subalgebra of all continuous functions. Choose two irrational numbers α and β such that 1, α and β are linearly independent over the rationals, define $a \in A$ by

$$a(s) = \begin{cases} 1/k & \text{if } s - k\alpha \in \mathbb{Z} \text{ for some } k \in \mathbb{Z} \setminus \{0\}, \\ 0 & \text{otherwise,} \end{cases}$$

and define $b \in A$ similarly to a (but using β instead of α). It is well know that the orbits of α and β are dense in Δ, and it is easy to see that the functions a and b are upper semi-continuous. Note that a is in the Glimm ideal A_s if $a(s) = 0$ (where $a(s)$ denotes here the value of a at s, not the coset in $A(s)$). Indeed, given $\varepsilon > 0$, there exists $\delta > 0$ such that $|a(t)| < \varepsilon$ for all t satisfying $|t - s| < \delta$. Choose $c \in C$ such that $0 \leq c \leq 1$, $c(s) = 0$ and $c(t) = 1$ if $|t - s| \geq \delta$. Then $\|a - ca\| < \varepsilon$, hence (letting $\varepsilon \to 0$) $a \in A_s$. Similarly, $b \in B_s$ if $b(s) = 0$.

Since for each $s \in \Delta$ at least one of $a(s)$ or $b(s)$ is 0, it follows that $(a + A_s) \otimes (b + A_s) = 0$ in $A(s) \otimes A(s)$ for all s, thus $\iota(a \otimes_C b) = 0$. On the other hand, since Ca and Cb are free $C-$modules (note that $ca = 0$ for $c \in C$ implies that $c = 0$ and similarly for b), the element $a \otimes_C b$ is a generator of the free $C-$module $Ca \odot_C Cb \cong C$. To see that $a \otimes_C b \neq 0$ in $A \odot_C A$, let $\hat{C}$ be the injective hull of C (which coincides as a $C-$module with the Johnson ring of quotients of C; see [22, 3.3.15]), let $\varphi_0 : Ca \to C \subseteq \hat{C}$ and $\psi_0 : Cb \to \hat{C}$ be defined by $\varphi_0(ca) = c$ and $\psi_0(cb) = c$ (resp.), and let $\varphi, \psi : A \to \hat{C}$ be extensions of φ_0 and ψ_0 as $C-$module homomorphisms. Defining $\rho : A \odot_C A \to \hat{C}$ by $\rho(x \otimes_C y) = \varphi(x)\psi(y)$, we see that $\rho(a \otimes_C b) = 1$, hence $a \otimes_C b \neq 0$.

In general, if A and B are abelian C^*-algebras containing C, then it is easy to see that for each possible C^*-norm α on $A \odot_C B$ the spectrum $\mathcal{P}$ of $A \otimes_C^\alpha B$ is homeomorphic with

$$\{(\varphi, \psi) \in P(A) \times P(B) : \ \varphi|C = \psi|C\},$$

where a pair (φ, ψ) corresponds to the character

$$(\varphi \otimes_C \psi)(a \otimes_C b) \stackrel{\text{def}}{=} \varphi(a)\psi(b).$$

(See [13] for the proof in the special case $C = \mathbb{C}$; the proof in general is almost the same.) Hence there is at most one C^*-norm on $A \odot_C B$. This C^*-(semi)norm is given of course by

$$\alpha(g) = \sup_{\rho \in \mathcal{P}} \rho(g). \tag{3.5}$$

It is easy to see that the quantity on the right side of (3.5) is equal to the norm of $\iota(g)$ for each $g \in A \odot_C B$, where ι is the $*-$homomorphism defined by (3.4). Since we have just seen that ι is not always injective, it follows that the quantity in (3.5) is not always a norm.

If Z is a von Neumann algebra contained in the centers of two C^*-algebras $A \subseteq L(E)$ and $B \subseteq L(F)$ (where E and F are faithful self-dual Hilbert $Z-$modules), $q: A \odot B \to A \odot_Z B$ is the quotient homomorphism and Θ and π are as in the diagram in the proof of Corollary 3.8, then $\pi\Theta q : A \odot B \to \bigoplus_{s\in\Delta} B(E(s) \otimes F(s))$ coincides with the restriction to $A \odot B$ of the direct sum over $(s \in \Delta)$ of maps $\pi_s^A \otimes \pi_s^B : A \otimes B \to B(E(s)) \otimes B(F(s))$ (where $\pi_s^A : A \to B(E(s))$ is the restriction of the representation of $L(E)$ on $E(s)$ defined immediately before Proposition 3.7, and π_s^B is defined similarly). Since π and Θ are isometric, it follows that q is bounded (when $A \odot B$ and $A \odot_Z B$ are equipped with the spatial norms) and extends uniquely to a $*-$homomorphism $\varphi : A \otimes B \to A \otimes_Z B$. Obviously φ is onto and the kernel of φ contains the closed linear span $[N]$ of the set $\{az \otimes b - a \otimes zb : \ a \in A, \ b \in B, \ z \in Z\}$. Thus, φ induces a $*-$epimorphism $\Phi : (A \otimes B)/[N] \to A \otimes_Z B$.

From now on denote

$$A \check{\otimes}_Z B = (A \otimes_Z B)/[N]$$

and

$$J_s = A_s \otimes B + A \otimes B_s.$$

Observe that $[N] \subseteq J_s$ for all $s \in \Delta$. Indeed, for each pure state ρ on $A \otimes B$ satisfying $\rho((Z \otimes Z)_{(s,s)}) = 0$ we have $\rho(ac \otimes b - a \otimes cb) = \rho(a \otimes b)\rho(c \otimes 1) - \rho(a \otimes b)\rho(1 \otimes c) = \rho(a \otimes b)(c(s) - c(s)) = 0$ for all $a \in A$, $b \in B$ and $c \in Z$ (see [13, 4.3.14]). Note that $Z \otimes Z$ is contained in the center of $A \otimes B$, the spectrum of $Z \otimes Z$ is $\Delta \times \Delta$ and the kernel of the character (s,t) is $Z_s \otimes Z + Z \otimes Z_t$ for all $s,t \in \Delta$ (by an application of the Stone - Weierstrass theorem). This implies that the Glimm ideal $(A \otimes B)_{(s,t)} = [(A \otimes B)(Z \otimes Z)_{(s,t)}]$ is equal to $A_s \otimes B + A \otimes B_t$; in particular $(A \otimes B)_{(s,s)} = J_s$. We need a simple lemma.

Lemma 3.10. *The $*-$homomorphism*

$$\Psi : A \check{\otimes}_Z B \to \bigoplus_{s\in\Delta} (A \otimes B)/J_s, \quad \Psi(a \otimes b + [N])(s) = a \otimes b + J_s$$

is one to one.

Proof. For each $g \in A \otimes B$ denote by $\dot{g}$ the coset of g in $A \check{\otimes}_Z B$ and by $g(s,t)$ the coset of g in $(A \otimes B)/(A \otimes B)_{(s,t)}$. Assume that $\dot{g} \in \ker \Psi$. We shall prove that $\|\dot{g}\| \leq \varepsilon$ for each $\varepsilon > 0$, hence $\dot{g} = 0$. Given $\varepsilon > 0$, let $W = \{(s,t) \in \Delta \times \Delta : \|g(s,t)\| \geq \varepsilon\}$. Since the function $(s,t) \mapsto \|g(s,t)\|$ is upper semicontinuous by [11], W is closed. Since $\dot{g} \in \ker \Psi$, $g(s,s) = 0$ for each $s \in \Delta$; hence there exists an open neighborhood $\mathcal{U}$ of the diagonal $\{(s,s) : \ s \in \Delta\}$ in $\Delta \times \Delta$ such that $\overline{\mathcal{U}} \cap W = \emptyset$. By Urysohn's lemma there exists $d \in Z \otimes Z$ such that $0 \leq d \leq 1$, $d|\overline{\mathcal{U}} = 0$ and $d|W = 1$. Then, with $g_0 = dg$, we have

$$\|g_0 - g\| \leq \varepsilon \tag{3.6}$$

and $g_0(s,t) = 0$ for all $(s,t) \in \mathcal{U}$.

Let $\{\mathcal{V}_i : \ i = 1, \ldots, n\}$ be a partition of the extremely disconnected compact space Δ into closed and open subsets such that $\mathcal{V}_i \times \mathcal{V}_i \subseteq \mathcal{U}$ for all i, and denote by p_i the projection in Z corresponding to $\mathcal{V}_i$. Since $g_0|\mathcal{U} = 0$, we have

$$g_0(s,t) = \sum_{i,j=1}^{n} p_i(s) g_0(s,t) p_j(t) = \sum_{i \neq j} p_i(s) g_0(s,t) p_j(t)$$

for all $s,t \in \Delta$; hence

$$g_0 = \sum_{i \neq j} (p_i \otimes 1) g_0 (1 \otimes p_j) = \sum_{i \neq j} (p_i \otimes 1 - 1 \otimes p_i) g_0 (1 \otimes p_j). \tag{3.7}$$

Approximating each term $g_0(1 \otimes p_j)$ by a sum of simple tensors, we see from (3.7) that $g_0 \in [N]$; hence we conclude from (3.6) that $\|\dot{g}\| \leq \varepsilon$. □

Corollary 3.11. *If $A_0 \subseteq A$ and $B_0 \subseteq B$ are concrete C*−algebras containing Z in their centers, then the natural homomorphisms $A_0 \otimes_Z B_0 \to A \otimes_Z B$ and $A_0 \check{\otimes}_Z B_0 \to A \check{\otimes}_Z B$ are injective.*

Proof. This result follows immediately by application of Remark 3.6, Corollary 3.8, Lemma 3.10 and the fact that the usual spatial tensor product preserves inclusions. □

The injectivity of the second homomorphism in Corollary 3.11 means that the two closed ideals, $N(A_0, B_0)$ in $A_0 \otimes B_0$ and $N(A,B)$ in $A \otimes B$, generated by all elements of the form $w = c \otimes 1 - 1 \otimes c$ $(c \in Z)$, satisfy $N(A_0,B_0) = N(A,B) \cap (A_0 \otimes B_0)$. The last equality can be proved easily also by using the fact that a left ideal in a C*−algebra is equal to the intersection of the left kernels of all (pure) states annihilating it (and that states can be extended to larger C*−algebras), which shows that the result is true also if Z is just a unital abelian C*−algebra (not necessarily a W*−algebra).

By (3.4) and Corollary 3.8 we have a natural $*$−homomorphism

$$\iota : A \otimes_Z B \to \bigoplus_{s \in \Delta} A(s) \otimes B(s)$$

and obviously $\Phi : A \check{\otimes}_Z B \to A \otimes_Z B$ must be one to one if the composition $\Gamma = \iota\Phi$ is one to one. For each $g \in A \otimes B$ let $g[s,t]$ denote the image of g in $A(s) \otimes B(t)$ under the tensor product of the two quotient maps. If the function $(s,t) \mapsto \|g[s,t]\|$ is continuous on $\Delta \times \Delta$ for each $g \in A \otimes B$, then the argument from the proof of Lemma 3.10 shows that Γ is one to one. If A or B is exact, then by [15, 4.6, 2.9] the function $\|g[\cdot,\cdot]\|$ is indeed continuous. (A C*−algebra A is called exact, if for each exact sequence of C*−algebras $0 \to J \to B \to B/J \to 0$ the sequence $0 \to A \otimes J \to A \otimes B \to A \otimes B/J \to 0$ is exact. This class is large, it includes for example all nuclear C*−algebras.) This proves the following theorem.

Theorem 3.12. *If A or B is exact, then the natural $*$−epimorphism $\Phi : A \check{\otimes}_Z B \to A \otimes_Z B$ is one to one.*

The author was not able to prove Theorem 3.12 without the assumption of exactness. Since both tensor products $\check{\otimes}_Z$ and $\otimes_Z$ preserve inclusions by Corollary 3.11, it would suffice to study the case when A and B are type I von Neumann algebras containing Z in their centers (see also [10, Remark 3.5]).

4. The spatial norm and Z−valued states

Based on the work of Halpern [12], it would be not hard to build a theory of Z−valued functionals and then to extend the classical results of Takesaki [25] (for example, the minimality of the spatial norm) to our present context. But since such an extension has been recently achieved in [4] and [10] (in a different way) in the setting when the role of Z is played by a general abelian C*−algebra, we shall limit our discussion here to results which are more specific for the setting where Z is an abelian W$^*-$algebra.

A $Z-$state on A is a positive $Z-$module homomorphism ω from A to Z such that $\omega(1)=1$ (hence $\|\omega\|=1$). The set of all $Z-$states on A will be denoted by $S_Z(A)$ (while $S(A)$ denotes the usual state space of A). If $A\subseteq L(E)$ for some Hilbert $Z-$module E, then each $x\in E$ with $\langle x,x\rangle=1$ defines a *vector $Z-$state* ω_x on A by $\omega_x(a)=\langle x,ax\rangle$. The space of all bounded $Z-$module homomorphisms from A to Z, denoted by $A^\natural$, is a $Z-$bimodule by $(z\omega)(a)=(\omega z)(a)\stackrel{\text{def}}{=}\omega(za)$ ($z\in Z$). The *weak* topology on bounded sets of $A^\natural$* is defined by the family of seminorms $\omega\mapsto|\nu(\omega(a))|$ ($a\in A,\ \nu\in Z_\sharp$), where $Z_\sharp$ is the predual of Z. ($A^\natural$ is the dual of the projective tensor product $A\hat{\otimes}_Z Z_\sharp$, but we shall not need this fact.) The unit ball of $A^\natural$ and its weak* closed subset $S_Z(A)$ are weak* compact.

Given a $Z-$module M, a subset $S\subseteq M$ is *$Z-$convex* if $cx+(1-c)y\in S$ for all $x,y\in S$ and all $c\in Z$ with $0\le c\le 1$. A point $v\in S$ is *$Z-$extreme* if the condition $v=cx+(1-c)y$, where $x,y\in S$ and $c\in Z$ with $0\le c\le 1$ and c, $1-c$ invertible, implies that $x=y=v$. The set of all $Z-$extreme points of $S_Z(A)$ (*pure $Z-$states on A*) is denoted by $P_Z(A)$. Here is a variant of the Krein-Milman theorem.

Theorem 4.1. *If K is a $Z-$convex weak* compact set of positive contractions in $A^\natural$, then K is the weak* closure of the smallest $Z-$convex set containing the $Z-$extreme points of K. In particular, $S_Z(A)$ is the weak* closure of the $Z-$convex hull of $P_Z(A)$.*

In the case A is a von Neumann algebra and Z the center of A, Theorem 4.1 was proved by Halpern in [12, 2.2]; since essentially the same arguments apply to the more general case where Z is contained in the center of a C$^*-$algebra A, we shall omit the proof (the result will not be important for what follows in this paper).

As it may be expected, in the case $A\subseteq L(E)$, $S_Z(A)$ is equal to the weak* closure of the $Z-$convex hull of the set of all vector $Z-$states. This is an immediate consequence of the following result.

Theorem 4.2. *If S is a subset of $S_Z(A)$ such that $\sup_{\omega\in S}\|\omega(a)\|=\|a\|$ for each self-adjoint $a\in A$, then $S_Z(A)=\overline{\mathrm{co}_Z S}$, where $\overline{\mathrm{co}_Z S}$ is the weak* closure of the $Z-$convex hull of S.*

Proof. Suppose the contrary, that $\rho\notin\overline{\mathrm{co}_Z S}$ for some $\rho\in S_Z(A)$, and put $S_0=\overline{\mathrm{co}_Z S}$. Then by the Hahn - Banach theorem there exists a linear functional θ on $A^\natural$ of the form

$$\theta(\omega)=\sum_{i=1}^{m}\nu_i(\omega(a_i))\quad(\omega\in A^\natural),\tag{4.1}$$

where $a_i\in A$ and $\nu_i\in Z_\sharp$ are fixed, such that

$$\sup_{\omega\in S_0}\mathrm{Re}(\theta(\omega))<\mathrm{Re}(\theta(\rho)).\tag{4.2}$$

Put $\nu=\sum_{i=1}^m|\nu_i|$ ($\in Z_\sharp$). Note that for each i there exists $c_i\in Z$ such that $\nu_i=\nu c_i$, and put $a=\sum_{i=1}^m c_ia_i$. Since each $\omega\in A^\natural$ is a homomorphism of $Z-$modules, we have from (4.1) that $\theta(\omega)=\nu(\omega(a))$; hence (4.2) can be rewritten as

$$\sup_{\omega\in S_0}\mathrm{Re}(\nu(\omega(a)))<\mathrm{Re}(\nu(\rho(a))).\tag{4.3}$$

Replacing a by $(1/2)(a+a^*)$ in (4.3), we may assume that $a^* = a$. Observe that the subset $Z_0 = \{\omega(a): \ \omega \in S_0\}$ of Z is an increasing net, that is, the maximum of two (and hence of finitely many) elements f and g of Z_0 is in Z_0. (Indeed, let $\mathcal{U} = \{t \in \Delta: \ f(t) > g(t)\}$ and let p be the characteristic function of the closure of the open set $\mathcal{U}$. Then p is a projection in Z (since Δ is extremely disconnected) and $\max\{f,g\} = pf + p^{\perp}g$. If $f = \omega_1(a)$ and $g = \omega_2(a)$, with $\omega_1, \omega_2 \in S_0$, then $\max\{f,g\} = \omega(a)$, where $\omega = p\omega_1 + p^{\perp}\omega_2 \in S_0$.) Denoting

$$c = \sup_{\omega \in S_0} \omega(a),$$

it follows now from (4.3), since ν is normal, that

$$\nu(c) < \nu(\rho(a)). \tag{4.4}$$

Replacing in (4.4) a by $a + t1$ (and consequently c by $c + t1$) for a suitable real t, we may further assume that a is positive.

We claim that there exists a projection $p_1 \in Z$ dominated by the support projection $p_0 \in Z$ of ν such that

$$\|cp_1\| < \|\rho(a)p_1\|. \tag{4.5}$$

Otherwise $\|\rho(a)p\| \leq \|cp\|$ for each projection $p \leq p_0$ in Z; hence

$$\rho(a)(s)p_0(s) \leq c(s)p_0(s)$$

for each s in the spectrum Δ of Z, and consequently $\rho(a)p_0 \leq cp_0$. But, since ν is positive, this would imply that $\nu(\rho(a)) \leq \nu(c)$, which would contradict (4.4).

By the hypothesis of the lemma and the definition of c we have that

$$\|ap\| = \sup_{\omega \in S} \|\omega(ap)\| \leq \sup_{\omega \in S_0} \|\omega(a)p\| = \|cp\|$$

for each projection $p \in Z$. Therefore (4.5) implies that $\|ap_1\| \leq \|cp_1\| < \|\rho(ap_1)\|$, which is a contradiction since $\|\rho\| = 1$ for each $\rho \in S_Z(A)$. $\square$

Given vector $Z-$states ω_x on A and ω_y on B, the vector $Z-$state $\omega_{x \otimes_Z y}$ is clearly the unique $Z-$state τ on $A \otimes_Z B$ satisfying $\tau(a \otimes_Z b) = \omega_x(a)\omega_y(b)$ for all $a \in A$ and $b \in B$. Since Theorem 4.2 implies that the $Z-$convex hull of vector $Z-$states is weak* dense in the space of all $Z-$states, it follows as in the classical case $Z = \mathbb{C}$ (see [13, 11.1.1]) that for arbitrary $\omega \in S_Z(A)$ and $\rho \in S_Z(B)$ there is a unique $\tau \in S_Z(A \otimes_Z B)$ satisfying $\tau(a \otimes_Z b) = \omega(a)\rho(b)$ for all $a \in A$ and $b \in B$. This state τ, called *the product $Z-$state*, will be denoted by $\omega \otimes_Z \rho$.

The *s-topology* on a self-dual Hilbert module E over a von Neumann algebra R is defined by the family of seminorms

$$x \mapsto (\rho(\langle x, x\rangle))^{1/2},$$

where ρ is a normal state on R. In a concrete representation of $E \subseteq \Lambda(E)$ (Theorem 2.1) the s-topology coincides on bounded sets with the usual strong operator topology. Hence, s-continuous linear functionals are weak* continuous and the same subspaces of E are closed in both topologies. (This was proved in a different way in [2].) Note also that the operators in $L(E)$ are weak* (and s-) continuous on E.

Remark 4.3. (i) Let F be an $R-$submodule of a self-dual Hilbert $R-$module E (where R is a W$^*-$algebra) and let $\overline{F}$ be the weak* closure of F. Then an application of the Kaplansky density theorem in $\Lambda(E)$ shows that the unit ball of F is dense in the unit ball of $\overline{F}$ in the weak* and in the s-topology.

(ii) The weak* closed submodules of a self-dual Hilbert $R-$module E are precisely the submodules of the form pE, where p is a projection in $L(E)$. This was observed in [2, p. 205] and can be proved also by considering the weak* closed right ideal in $\Lambda(E)$ generated by a submodule of E, which is well known to be generated by a projection in $\Lambda(E)$.

Theorem 4.4. *Let E and F be faithful self-dual Hilbert $Z-$modules and $A \subseteq L(E)$, $B \subseteq L(F)$ two C^*-algebras containing (an isomorphic copy of) Z. Then the spatial norm of each $g \in A \odot_Z B$ can be expressed as*

$$\|g\|^2 = \sup_{\tau,h} \frac{\|\tau(h^*g^*gh)\|}{\|\tau(h^*h)\|}, \tag{4.6}$$

where the supremum is over all elements $h \in A \odot_Z B$ and all product $Z-$states τ on $A \otimes_Z B$ such that $\tau(h^*h) \neq 0$.

Proof. Suppose first that E and F are cyclic in the sense that $E = [Ax]$ and $F = [By]$ for some unit vectors $x \in E$ and $y \in F$. Here $[\mathcal{S}]$ denotes the s-closed linear span of a subset S of E or F. Then the unit balls of $E_0 \stackrel{\text{def}}{=} Ax$ and $F_0 \stackrel{\text{def}}{=} By$ are s-dense in the unit balls of E and F (respectively) by Remark 4.3(i); hence, given $u \in E$ and $v \in F$, there exist bounded nets (u_k) in E_0 and (v_k) in F_0 converging to u and v (resp.) in the s-topology. Then the net $(u_k \otimes_Z v_k)$ converges to $u \otimes_Z v$ in the s-topology of $E\overline{\otimes}_Z F$. To see this, write

$$u_k \otimes_Z v_k - u \otimes_Z v = (u_k - u) \otimes_Z v_k + u \otimes_Z (v_k - v)$$

and note that a net $(x_k \otimes_Z y_k)$ converges to 0 in the s-topology whenever the nets (x_k) and (y_k) are bounded and one of them, say (x_k), converges to 0, since for each normal state ν on Z we have

$$(\nu(\langle x_k \otimes_Z y_k, x_k \otimes_Z y_k \rangle))^{1/2} \leq \|y_k\|(\nu(\langle x_k, x_k \rangle))^{1/2} \to 0.$$

This shows that $[E_0 \otimes_Z F_0]$ contains all vectors of the form $u \otimes_Z v$, hence $[E_0 \otimes_Z F_0] = E\overline{\otimes}_Z F$, so by Remark 4.3(i) the unit ball of $E_0 \otimes_Z F_0$ is dense in the unit ball of $E\overline{\otimes}_Z F$ in the s-topology. Therefore for each $g \in A \odot_Z B$ we have

$$\begin{aligned} \|g\|^2 &= \sup\{\tfrac{\|g(w)\|^2}{\|w\|^2} :\ w \in E_0 \otimes_Z F_0,\ w \neq 0\} \\ &= \sup\{\tfrac{\|gh(x\otimes_Z y)\|^2}{\|h(x\otimes_Z y)\|^2} :\ h \in A \odot_Z B,\ h(x \otimes_Z y) \neq 0\} \\ &= \sup\{\tfrac{\|\tau_0(h^*g^*gh)\|}{\|\tau_0(h^*h)\|} :\ h \in A \odot_Z B,\ \tau_0(h^*h) \neq 0\}, \end{aligned}$$

where $\tau_0 = \omega_x \otimes_Z \omega_y$. This implies that $\|g\|^2$ is dominated by the right side of (4.6), but the reverse inequality is obvious.

In general, a standard maximality argument (using Remark 4.3(ii)) shows that E and F are orthogonal sums of s-closed cyclic submodules, and the theorem follows by the same reasoning as in the special case $Z = \mathbb{C}$ (see [13, 11.1.2]). ∎

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